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TEOREMA LUI ROLLE

Enunt: Fie f:[a,b]→R ,a<b o functie continua pe intervalul inchis [a,b],derivabila pe intervalul deschis (a,b) si astfel incat f(a)=f(b).Atunci exista cel putin un punct c∈(a,b) astfel incat f ’(c)=0.

Demonstratie:

$$\left.\begin{matrix} f \text{ cont pe } [a,b] \\ [a,b] \text{ - compact} \end{matrix}\right\} f \text{ marginita si isi atinge marginile in } [a,b]$$

Fie m=inf f(x)

M=sup f(x)   x∈[a,b]

Cazul I

M>f(a)=f(b)

$$\left.\begin{matrix} (\exists)\, c \in [a,b] \text{ a.i } M = f(c) \\ c \neq a, c \neq b \end{matrix}\right\} \Rightarrow c \in (a,b)$$

$$\left.\begin{matrix} c \in (a,b) \\ c \text{ maxim local} \end{matrix}\right\} \Rightarrow (T.Fermat)\, f'(c) = 0$$

Cazul II

$m < f(a) = f(b)$

$$\left.\begin{matrix} (\exists)\, c \in [a,b] \text{ a.i } m = f(c) \\ c \neq a, c \neq b \end{matrix}\right\} \Rightarrow c \in (a,b)$$

$$\left.\begin{array}{r} c \in (a,b) \\ c \text{ minim local} \end{array}\right\} \Rightarrow (T.Fermat)\ f\,'(c) = 0$$

Cazul III

$m = M \Rightarrow f$ constanta pe $[a,b] \Rightarrow f\,'(c) = 0$

COROLAR : Intre doua zerouri ale unei functii derivabile pe un interval se afla cel putin un zerou al derivatei.

Dem :

Fie $f : I \rightarrow R \qquad I \subseteq R$

$x_1, x_2 \in I$ a.i $f(x_1) = f(x_2) = 0$

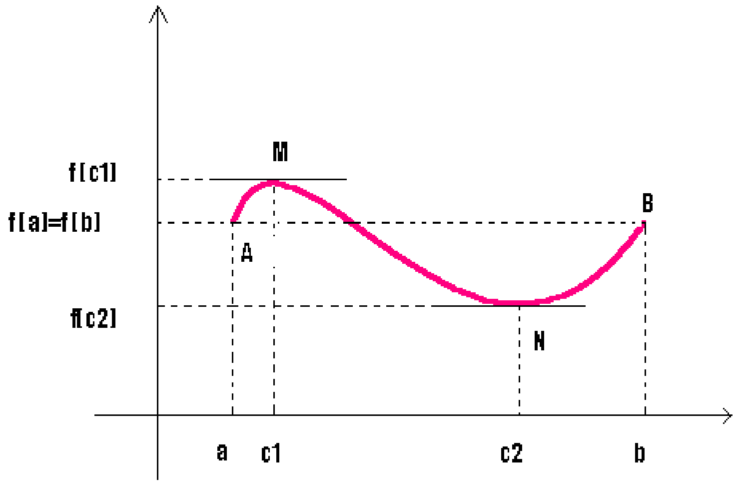
$$f : (x_1, x_2) \rightarrow R$$

$$\left.\begin{array}{l} f \text{ derivabila pe } (x_1, x_2) \\ f(x_1) = f(x_2) \end{array}\right\}$$

$\Rightarrow$ (T.Rolle) $(\exists)$ cel putin un $c \in (x_1, x_2)$ a.i $f\,'(c) = 0$

INTERPRETARE GEOMETRICA

Tangentele duse prin punctele "c" corespunzatoare graficului functiei sunt paralele cu Ox si cu dreapta AB.



OBS! Toate conditiile din teorema lui Rolle sunt necesare. Daca se renunta la una din conditii atunci teorema nu mai este valabila.

APLICATII

Sa se studieze aplicabilitatea T.Rolle pentru functiile:
1)f:[-1,1]→R

$$f(x) = \begin{cases} x^2 + 1 & \text{x} \in [\text{-1,0}) \\ x + 1 & \text{x} \in [0,1] \end{cases}$$

continuitate

$$ls = \lim_{x \uparrow 0} f(x) = \lim_{x \uparrow 0} (x^2 + 1) = 1$$

$$ld=\lim_{x\downarrow 0} f(x)=\lim_{x\downarrow 0}(x+1)=1$$

$$ls=ld=1\Rightarrow f \text{ continua in x} = 0$$

f continua pe[-1,1]

derivabilitate

$$\text{fs}=\lim_{x\uparrow 0}\frac{f(x)-f(0)}{x}=\lim_{x\uparrow 0}\frac{x^2+1-1}{x}=\lim_{x\uparrow 0}x=0$$

$$fd=\lim_{x\downarrow 0}\frac{f(x)-f(0)}{x}=\lim_{x\uparrow 0}\frac{x+1-1}{x}=\lim_{x\uparrow 0}1=0$$

$$fs\neq fd\Rightarrow f \text{ nu este derivabila in x} = 0$$

f nu satisface T lui Rolle

$$\text{2)f} : [0,\frac{\Pi}{2}]\rightarrow \text{R}$$

$$\text{f(x)} = \begin{cases} \text{tg x, x} \in [0,\frac{\Pi}{4}) \\ ctg\ x, x\in [\frac{\Pi}{4},\frac{\Pi}{2}] \end{cases}$$

continuitate

$$ls=\lim_{x\uparrow\frac{\Pi}{4}} f(x)=\lim_{x\uparrow\frac{\Pi}{4}} tg \text{ x} = 1$$

$$\text{ld}=\lim_{x\downarrow\frac{\Pi}{4}} f(x)=\lim_{x\downarrow\frac{\Pi}{4}} ctg\ x = 1$$

derivabilitate

$$ls = \lim_{x \uparrow \frac{\Pi}{4}} \frac{f(x) - f(\frac{\Pi}{4})}{x - \frac{\Pi}{4}} = \lim_{x \uparrow \frac{\Pi}{4}} \frac{tg\ \mathrm{x} - 1}{\mathrm{x} - \frac{\Pi}{4}} = \lim_{x \uparrow \frac{\Pi}{4}} \frac{tg\ \mathrm{x} - \mathrm{tg}\frac{\Pi}{4}}{\mathrm{x} - \frac{\Pi}{4}} =$$

$$= \lim_{x \uparrow \frac{\Pi}{4}} \frac{\frac{\sin(x - \frac{\Pi}{4})}{\cos x \cos\frac{\Pi}{4}}}{\mathrm{x} - \frac{\Pi}{4}} = \lim_{x \uparrow \frac{\Pi}{4}} \frac{\sin(x - \frac{\Pi}{4})}{\mathrm{x} - \frac{\Pi}{4}} \lim_{x \uparrow \frac{\Pi}{4}} \frac{1}{\cos x \cos\frac{\Pi}{4}} =$$

$$= \lim_{x \uparrow \frac{\Pi}{4}} \frac{1}{\cos x \cos\frac{\Pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}} = 2$$

$$fd = \lim_{x \downarrow \frac{\Pi}{4}} \frac{f(x) - f(\frac{\Pi}{4})}{x - \frac{\Pi}{4}} = \lim_{x \downarrow \frac{\Pi}{4}} \frac{ctgx - 1}{x - \frac{\Pi}{4}} = \lim_{x \downarrow \frac{\Pi}{4}} \frac{ctgx - ctg\frac{\Pi}{4}}{x - \frac{\Pi}{4}} =$$

$$= \lim_{x \downarrow \frac{\Pi}{4}} \frac{\frac{\sin(\frac{\Pi}{4} - x)}{\sin x \sin\frac{\Pi}{4}}}{x - \frac{\Pi}{4}} = \lim_{x \downarrow \frac{\Pi}{4}} \frac{\sin(\frac{\Pi}{4} - x)}{-(\frac{\Pi}{4} - x)} \lim_{x \downarrow \frac{\Pi}{4}} \frac{1}{\sin x \sin\frac{\Pi}{4}} =$$

$$= -\lim_{x \downarrow \frac{\Pi}{4}} \frac{1}{\sin x \sin\frac{\Pi}{4}} = -\frac{1}{\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}} = -2$$

$fs \neq fd \Rightarrow$ f nu este derivabila in $x = \frac{\Pi}{4}$

f nu satisface T lui Rolle

3)Fie functia $f(x) = (ax+b)^m(cx+d)^n, m>1, n>1, x \in R$. Sa se demonstreze ca $\frac{mad+nbc}{ac(m+n)}$ este cuprins intre $\frac{b}{a}$ si $\frac{d}{c}$.

$$f'(x) = m(ax+b)^{m-1}a(cx+d)^n + n(cx+d)^{n-1}c(ax+b)^m$$

$$f'(x) = (ax+b)^{m-1}(cx+d)^{n-1}[macx+mad+ncax+ncb]$$

$$f'(x) = (ax+b)^{m-1}(cx+d)^{n-1}[(mac+nca)x+mad+ncb]$$

$$f'(x) = 0$$

$$\left.\begin{array}{l} \Rightarrow (ax+b)^{m-1} = 0 \Rightarrow x = -\frac{b}{a} \\ sau\ (cx+d)^{n-1} = 0 \Rightarrow x = -\frac{d}{c} \end{array}\right\}$$

$\Rightarrow$ *nu* apartin intervalului considerat

$$x = -\frac{mad+nbc}{mac+nca}$$

$$f(x) = 0$$

$$\Rightarrow (ax+b)^m = 0 \Rightarrow x = -\frac{b}{a}$$

$$(cx+d)^n = 0 \Rightarrow x = -\frac{d}{c}$$

$\Rightarrow$ (*conform* corolarului lui Rolle)

$$\frac{b}{a} < \frac{mad + nbc}{ac(m+n)} < \frac{d}{c} \text{ sau}$$

$$\frac{\mathrm{d}}{\mathrm{c}} < \frac{mad + nbc}{ac(m+n)} < \frac{b}{a}$$

dupa cum $\frac{\mathrm{b}}{\mathrm{a}} < \frac{d}{c}$ sau $\frac{\mathrm{d}}{\mathrm{c}} < \frac{b}{a}$

4)*Fie* functia :

$$f(x) = \begin{cases} x^2 + mx + n & , x \in [-1,0) \\ p\,x^2 + 4x + 4 & , x \in [0,1] \end{cases}$$

definita pe $[-1,1]$, $m, n, p \in \mathbb{R}$. Sa se determine parametrii m, n, p astfel incat functiei date sa i se poata aplica teorema lui Rolle pe intervalul $[-1,1]$.

continuitate

$$ls = \lim_{x \uparrow 0} f(x) = \lim_{x \uparrow 0} (x^2 + mx + n) = n$$

$$ld = \lim_{x \downarrow 0} f(x) = \lim_{x \downarrow 0} (p\,x^2 + 4x + 4) = 4$$

derivabilitate

$$fs = \lim_{x \uparrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \uparrow 0} \frac{x^2 + mx + n - 4}{x} = \lim_{x \uparrow 0} \frac{x^2 + mx + 4 - 4}{x}$$

$$= m$$

$$fd = \lim_{x\uparrow 0} \frac{f(x) - f(0)}{x} = \lim_{x\uparrow 0} \frac{px^2 + 4x + 4 - 4}{x} = 0$$

f satisface T lui Rolle $\Rightarrow$

$ls = ld \Rightarrow n = 4$

$fs = fd \Rightarrow m = 4$

$f(-1) = f(1) \Rightarrow 1 - m + n = p + 8$

$1 - 4 + 4 = p + 8$

$p = 7$

5)Se considera functia $f : [0,1] \to R$:

$$f(x) = \begin{cases} x\sin\frac{\Pi}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

Aplicand lui f teorema lui Rolle pe fiecare interval $[\frac{1}{n+1}, \frac{1}{n}]$ $n \geq 1$ intreg, sa se arate ca ecuatia $tgx = x$ are solutii pe fiecare interval $(n\Pi, (n+1)\Pi)$.

$$\left.\begin{array}{l} f(\frac{1}{n+1}) = \frac{1}{n+1}\sin\Pi(n+1) = 0 \\ f(\frac{1}{n}) = \frac{1}{n}\sin\Pi n = 0 \end{array}\right\}$$

$$\Rightarrow (\exists)\ \xi \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \text{ a.i } f\,'(\xi) = 0$$

$$f\,'(x) = \sin\frac{\Pi}{x} + x\cos\frac{\Pi}{x}$$

$$f\,'(\xi) = 0$$

$$\Rightarrow \sin\frac{\Pi}{\xi} + x\cos\frac{\Pi}{\xi} = 0 \quad |:\cos\frac{\Pi}{\xi}$$

$$\text{tg}\frac{\Pi}{\xi} = \frac{\Pi}{\xi} \Rightarrow \frac{\Pi}{\xi} \text{ solutie a ecuatiei tgx} = x$$

$$\frac{1}{n+1} < \xi < \frac{1}{n}$$

$$n < \frac{1}{\xi} < n+1 \ |\Pi$$

$$n\Pi < \frac{\Pi}{\xi} < \Pi(n+1) \Rightarrow \frac{\Pi}{\xi} \in (n\Pi, \Pi(n+1))$$

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