# Permutari

1.Notiunea de permutare.

Fie A o multime finita de „n“ elemente, adica A={1, 2, 3, …, n}.

O functie bijectiva σ:A→A se numeste permutare (substitutie) de gradul n.

P:Numarul tuturor permutarilor de ordin n este egal cu n! .

2.Produsul (compunerea) permutarilor.

Fie σ si τ doua permutari de acelasi grad n.
Prin compunerea celor doua permutari se intelege o noua permutare σ oτ :A→A cu prop. (σ oτ)(k)=σ(τ(k)).

3.Proprietati ale compunerii permutarilor.

P1: Asociativitatea compunerii

(σoτ)oφ=σo(τoφ), oricare ar fi σ;τ;φ ϵ $S_n$.

P2: Compunerea permutarilor nu este comutativa

σoτ=τoσ

P3: Element neutru

σoe=eoσ  oricare ar fi σ ϵ $S_n$

e(i)=i →permutarea identica

P4: Element simetrizabil

σoσ=σoσ=e

4.Transpozitii.

Se numeste transpozitie o permutare de forma σ(i,j) sau (i,j) cu proprietatea

$$\sigma_{ij}(k) = \begin{cases} j : k = i \\ i : k = j \\ k : k \neq i, j \end{cases}$$

Proprietati:

P1: $\sigma^2_{ij} = e$

P2: $\sigma_{ij} = \sigma_{ij}$

P3: $\sigma_{ij} = \sigma_{ji}$

Numarul tuturor transpozitiilor de ordin n este egal cu $C_n^2$.

Numarul tuturor transpozitiilor de ordin n este egal cu numarul perechilor (i,j) cu proprietatea ca i<j<n.

5.Inversiunile unei permutari.

Se numeste inversiune intr-o permutare σ o pereche de elemente (i,j) i<j cu proprietatea ca σ(i)> σ(j).

Numarul inversiunilor intr-o permutare se noteaza cu M(σ) <= $C_n^2$.

6.Signatura unei permutari.

Fie σ∈ $S_n$. Numarul □(σ) =(-1) se numeste signatura (semnul) permutarii σ.

$$\varepsilon(\sigma) = (-1)^{M(\sigma)}$$

ε (σ) = 1 daca M(σ) este par

-1 daca M(σ) este impar

*$\sigma$ se numeste permutare para daca are un numar par de inversiuni.

*$\sigma$ se numeste permutare impara daca are un numar impar de inversiuni.

Teorema 1. Orice transpozitie este o permutare impara.

Teorema 2. Daca $\sigma \in S_n$ atunci $\varepsilon(\sigma) = \prod (\sigma(i)-\sigma(j))/(i-j)$.

Teorema 3. Daca $\sigma,\tau \in S_n$ atunci $\varepsilon(\sigma o \tau) = \varepsilon(\sigma) o \varepsilon(\tau)$.

Teorema 4. Daca $\sigma \in S_n$ este o permutare atunci $\sigma$ poate fi descompusa ca produs de transpozitii.

Obs: Daca $\sigma$ este para ea poate fi descompusa ca produs par de transpozitii si daca este impara ea poate fi descompusa ca produs impar de transpozitii.

Aplicatii.

1. Fie permutarile $\sigma=\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}$ si $\tau=\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix}$ . Sa se calculeze

$\sigma o \tau$ si $\tau o \sigma$.

$\sigma o \tau=\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 4 & 1\end{pmatrix}$ $\tau o \sigma=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{pmatrix}$

2. Sa se determine numarul de inversiuni si signatura pentru fiecare dintre permutarile urmatoare:

* 1 2 3
  2 3 1

$M(\sigma) = 2 \Rightarrow \varepsilon(\sigma) = 1$

* 1 2 3 4
  2 4 1 3

$M(\sigma) = 3 \Rightarrow \varepsilon(\sigma) = -1$

* 1 2 3 4
  4 1 2 3

$M(\sigma) = 3 \Rightarrow \varepsilon(\sigma) = -1$

* 1 2 3 4 5
  5 3 4 1 2

$M(\sigma) = 8 \Rightarrow \varepsilon(\sigma) = 1$

3. Fie permutarea $\sigma = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{matrix}$ . Sa se scrie $\sigma$ ca produs de transpozitii. Aceeasi problema pentru permutarea $\tau = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{matrix}$ .

$$*(4,5) \circ \sigma = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{matrix} \circ \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{matrix} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{matrix} = \sigma_1$$

$$(1,3) \circ \sigma_1 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{matrix} \circ \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{matrix} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{matrix} = \sigma_2$$

$$(2,3) \circ \sigma_2 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{matrix} \circ \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{matrix} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{matrix} = e$$

⇨ $\sigma = (4,5) \circ (1,3) \circ (2,3)$

$$*(1,6) \circ \tau = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 3 & 4 & 5 & 1 \end{matrix} \circ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{matrix} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 3 & 2 & 6 \end{matrix} = \tau_1$$

$$(2,5) \circ \tau_1 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \circ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} = \tau_2$$

$$(3,4)\circ\tau_2 = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&2&4&3&5&6 \end{pmatrix} \circ \begin{pmatrix} 1&2&3&4&5&6 \\ 1&4&2&3&5&6 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6 \\ 1&3&2&4&5&6 \end{pmatrix} = \tau_3$$

(upper row above the permutations: 1 5 3 4 2 6    1 4 5 3 2 6    1 4 2 3 5 6)

$(2,3)\circ\tau_3 = e$

⇨ $\tau = (1,6)\circ(2,5)\circ(3,4)\circ(2,3)$.

4. Fie permutarea $\sigma \in S_{2n}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \ldots & n & n+1 & n+2 \ldots & 2n \\ 1 & 3 & 5 & 7 \ldots & 2n-1 & 2 & 4 \ldots & 2n \end{pmatrix}.$$

Sa se determine numarul inversiunilor permutarii σ.

Sa se determine „n“ astfel incit σ sa fie para (respectiv impara).

$$\varepsilon(\sigma) = (-1)^{n(n-1)/2}$$

$*\, n = 4k$

$$\Rightarrow \varepsilon(\sigma) = (-1)^{4k(4k-1)/2} = 1$$

$*\, n = 4k+1$

$$\Rightarrow \varepsilon(\sigma) = (-1)^{4k(4k+1)/2} = 1$$

$*\, n = 4k+2$

$$\Rightarrow \varepsilon(\sigma) = (-1)^{(4k+1)(4k+2)/2} = -1$$

$*\, n = 4k+3$

$$\Rightarrow \varepsilon(\sigma) = (-1)^{(4k+2)(4k+3)/2} = -1$$

$M(\sigma) = 1+2+3+\ldots+ n-1 = n(n-1)/2$

5. Sa se determine numarul inversiunilor permutarii σ.

$$\begin{pmatrix} 1234 \ldots n & n+1 & n+2 & n+3 \ldots 2n \\ 2468 \ldots 2n & 1 & 3 & 5 \ldots 2n-1 \end{pmatrix}$$

$M(\sigma)=1+2+3+4+ \dots +n = n(n+1)/2$

6. Determinati $\sigma \in S_7$ astfel incit

$$\begin{pmatrix}1234567\\7321654\end{pmatrix} o\, \sigma\, o \begin{pmatrix}1234567\\3127456\end{pmatrix} = \begin{pmatrix}1234567\\5136472\end{pmatrix}$$

$$\sigma1^{-1}| \sigma1 o \sigma o \sigma2 = \sigma3$$

$$(\sigma1^{-1} o\, \sigma1)\, o\, (\sigma\, o\, \sigma2) = \sigma1^{-1} o\, \sigma3$$

$$\sigma\, o\, \sigma2 = \sigma1^{-1} o\, \sigma3 \sigma2^{-1}$$

$$\sigma = \sigma1^{-1} o\, \sigma3\, o\, \sigma2^{-1}$$

$$\sigma = \begin{pmatrix}1234567\\4327651\end{pmatrix} o \begin{pmatrix}1234567\\5136472\end{pmatrix} o \begin{pmatrix}1234567\\2315674\end{pmatrix}$$

$$\sigma = \begin{pmatrix}1234567\\4267135\end{pmatrix}$$

7. Rezolvati in $S_5$ ecuatia:

$\sigma o X = X o \sigma \quad \sigma = \begin{matrix}1\ 2\ 3\ 4\ 5\\2\ 3\ 1\ 5\ 4\end{matrix}$

$X = \begin{matrix}1\ 2\ 3\ 4\ 5\\a\ b\ c\ d\ e\end{matrix}$

$X o \sigma = \begin{matrix}1\ 2\ 3\ 4\ 5\\a\ b\ c\ d\ e\end{matrix}\ o\ \begin{matrix}1\ 2\ 3\ 4\ 5\\2\ 3\ 1\ 5\ 4\end{matrix} = \begin{matrix}1\ 2\ 3\ 4\ 5\\b\ c\ a\ e\ d\end{matrix}$

$\sigma o X = \begin{matrix}1\ 2\ 3\ 4\ 5\\2\ 3\ 1\ 5\ 4\end{matrix}\ o\ \begin{matrix}1\ 2\ 3\ 4\ 5\\a\ b\ c\ d\ e\end{matrix} = \begin{matrix}1 & 2 & 3 & 4 & 5\\\sigma(a) & \sigma(b) & \sigma(c) & \sigma(d) & \sigma(e)\end{matrix}$

=> $\sigma(a) = b$

$\sigma(b) = c$

$\sigma(c) = a$

$\sigma(d) = e$

$\sigma(e) = d \Rightarrow d,e \in \{4,5\}$

CAZUL I: d=4
e=5

$\Rightarrow \sigma(a) = b$

$\sigma(b) = c$

$\sigma(c) = a$

i) <u>a=1</u> $\Rightarrow \sigma(1) = b$ dar $\sigma(1) = 2 \Rightarrow$ <u>b=2</u>

$\sigma(b) = c \Rightarrow \sigma(2) = c$ dar $\sigma(2) = 3 \Rightarrow$ <u>c=3</u>

$\sigma(c) = 1$

$\Rightarrow X_1 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{matrix}$

ii) <u>a=2</u> $\Rightarrow \sigma(2) = b$ dar $\sigma(2) = 3 \Rightarrow$ <u>b=3</u>

$\sigma(b) = c \Rightarrow \sigma(3) = c$ dar $\sigma(3) = 1 \Rightarrow$ <u>c=1</u>

$\sigma(c) = 2$

$\Rightarrow X_2 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{matrix}$

iii) <u>a=3</u> $\Rightarrow \sigma(3) = b$ dar $\sigma(3)=1 \Rightarrow$ <u>b=1</u>

$\sigma(b) = c \Rightarrow \sigma(1) = c$ dar $\sigma(1)=2 \Rightarrow$ <u>c=2</u>

$\Rightarrow X_3 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{matrix}$

CAZUL II: d=5
e=4

i) a=1

$\Rightarrow X_4 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{matrix}$

ii) a=2

=> $X_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$

iii) a=3

=> $X_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$.

8. Fie permutare $u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$. Sa se arate ca nu exista nici o permutare $X \in S_4$, astfel incit $X^2 = u$.

$\varepsilon(X^2) = 1$

$\varepsilon(u) = -1$ => nu exista X.

9. Fie permutarea $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & i & 3 & j & 1 \end{pmatrix}$. Sa se determine i si j astfel incit $\sigma$ sa fie o permutare para (respectiv impara).

i=2 sau i=5

j=5 j=2

*i=2 si j=5

=> $\varepsilon(\sigma) = -1$ => permutarea este impara

*i=5 si j=2

=> $\varepsilon(\sigma) = 1$ => permutare este para.

10. Se dau numerele reale strict pozitive $a_1 < a_2 < \ldots < a_n$.

$$S\sigma = \sum_{i=1}^{n} \frac{1}{ai * a\sigma(i)}$$

Pentru ce permutare $\sigma \in S_n$ suma

este maxima.

Fie $\tau = \sigma \circ (k,j)$

$$\sigma = \begin{pmatrix} 1 \ldots 2 \ldots 3 \ldots k \ldots j \ldots n \\ \sigma(1) \ldots \sigma(2) \ldots \sigma(3) \ldots \sigma(k) \ldots \sigma(j) \ldots \sigma(n) \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 \ldots 2 \ldots 3 \ldots k \ldots j \ldots n \\ \sigma(1) \ldots \sigma(2) \ldots \sigma(3) \ldots \sigma(j) \ldots \sigma(k) \ldots \sigma_n \end{pmatrix}$$

$$S\sigma - S' = \frac{1}{ak}\left(\frac{a\sigma(j) - a\sigma(k)}{a\sigma(j) * a\sigma(k)}\right) + \frac{1}{aj}\left(\frac{a\sigma(k) - a\sigma(j)}{a\sigma(j) * a\sigma(k)}\right)$$

$$S' = \sum_{i=1}^{n} \frac{1}{ai * a\tau(i)}$$

$$= \frac{(a\sigma(j) - a\sigma(k)) * (aj - ak)}{ak * aj * a\sigma(k) * a\sigma(j)} > 0$$

$$S\sigma - S' > 0 \Rightarrow S\sigma > S' (aratam)$$

$$\Rightarrow a\sigma(j) - a\sigma(k) > 0 \Rightarrow a\sigma(j) > a\sigma(k) \Rightarrow \sigma = e.$$

$$S\sigma - S' = \sum_{i=1}^{n} \frac{1}{ai * a\sigma(i)} - \sum_{i=1}^{n} \frac{1}{ai * a\tau(i)}$$

$$S\sigma - S' = \sum_{i=1}^{n} \frac{1}{ai * a\sigma(i)} + \frac{1}{ak * a\sigma(k)} + \frac{1}{aj * a\sigma(j)} -$$

$$- \sum_{i=1}^{n} \frac{1}{ai * a\tau(i)} - \frac{1}{ak * a\tau(k)} - \frac{1}{aj * a\tau(j)}$$

$$S\sigma - S' = \frac{1}{ak * a\sigma(k)} + \frac{1}{aj * a\sigma(j)} - \frac{1}{ak * a\tau(k)} - \frac{1}{aj * a\tau(j)}$$

$$S\sigma - S' = \frac{1}{ak}\left(\frac{1}{a\sigma(k)} - \frac{1}{a\sigma(j)}\right) + \frac{1}{aj}\left(\frac{1}{a\sigma(j)} - \frac{1}{a\sigma(k)}\right)$$

11. Se dau numerele reale strict pozitive $a_1 < a_2 < \ldots < a_n$. Pentru ce permutare $\sigma \in S_n$ produsul

$$P\sigma = \prod_{i=1}^{n}\left(ai^{r} + \frac{1}{a\sigma(i)^{s}}\right) \text{ este } \max im?$$

(r se s sunt doua numere naturale >=1).

$\tau = \sigma o(k,j)$

$$\sigma = \begin{pmatrix} 1 & 2 & 3\ldots & k\ldots & j\ldots & n \\ \sigma(1) & \sigma(2) & \sigma(3)\ldots & \sigma(k)\ldots & \sigma(j)\ldots & \sigma(n) \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3\ldots & k\ldots & j\ldots & n \\ \sigma(1) & \sigma(2) & \sigma(3)\ldots & \sigma(j)\ldots & \sigma(k)\ldots & \sigma(n) \end{pmatrix}$$

$$P\tau = \prod_{i=1}^{n}\left(ai^{r} + \frac{1}{a\tau(i)^{s}}\right)$$

$$P\sigma > P\tau (Aratam)$$

$$\Rightarrow \prod_{i=1}^{n}\left(ai^{r} + \frac{1}{a\sigma(i)^{s}}\right) * \left(ak^{r} + \frac{1}{a\sigma(k)^{s}}\right) * \left(aj^{r} + \frac{1}{a\sigma(j)^{s}}\right) >$$

$$\prod_{i=1}^{n}\left(ai^{r} + \frac{1}{a\sigma(i)^{s}}\right) * \left(ak^{r} + \frac{1}{a\sigma(j)^{s}}\right) * \left(aj^{r} + \frac{1}{a\sigma(k)^{s}}\right)$$

$$\Rightarrow \left(ak^{r} + \frac{1}{a\sigma(k)^{s}}\right) * \left(aj^{r} + \frac{1}{a\sigma(j)^{s}}\right) > \left(ak^{r} + \frac{1}{a\sigma(j)^{s}}\right) * \left(aj^{r} + \frac{1}{a\sigma(k)^{s}}\right)$$

$$\Rightarrow \left(ai^{r} - aj^{s}\right) * \left(a\sigma(j)^{r} - a\sigma(i)^{s}\right) \le 0 \Rightarrow \sigma(i) < \sigma(j)$$

$$\Rightarrow \sigma = e.$$

12. Pentru ce permutare $\sigma \in S_n$ suma

$$S\sigma = \sum_{i=1}^{n} \frac{1}{ai * a\sigma(i)}$$

este minima?

Fie τ =σo(k,j)

$$\sigma = \begin{pmatrix} 1 \ldots 2 \ldots 3 \ldots k \ldots j \ldots n \\ \sigma_{(1)} \ldots \sigma_{(2)} \ldots \sigma_{(3)} \ldots \sigma_{(k)} \ldots \sigma_{(j)} \ldots \sigma_{(n)} \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 \ldots 2 \ldots 3 \ldots k \ldots j \ldots n \\ \sigma_{(1)} \ldots \sigma_{(2)} \ldots \sigma_{(3)} \ldots \sigma_{(j)} \ldots \sigma_{(k)} \ldots \sigma_{n} \end{pmatrix}$$

$$S' = \sum_{i=1}^{n} \frac{1}{ai * a\tau(i)}$$

$$S\sigma - S' > 0 \Rightarrow S\sigma > S'(aratam)$$

$$S\sigma - S' = \sum_{i=1}^{n} \frac{1}{ai * a\sigma(i)} - \sum_{i=1}^{n} \frac{1}{ai * a\tau(i)}$$

$$S\sigma - S' = \sum_{i=1}^{n} \frac{1}{ai * a\sigma(i)} + \frac{1}{ak * a\sigma(k)} + \frac{1}{aj * a\sigma(j)} -$$

$$\ldots\ldots - \sum_{i=1}^{n} \frac{1}{ai * a\tau(i)} - \frac{1}{ak * a\tau(k)} - \frac{1}{aj * a\tau(j)}$$

$$S\sigma - S' = \frac{1}{ak * a\sigma(k)} + \frac{1}{aj * a\sigma(j)} - \frac{1}{ak * a\tau(k)} - \frac{1}{aj * a\tau(j)}$$

$$S\sigma - S' = \frac{1}{ak}\left( \frac{1}{a\sigma(k)} - \frac{1}{a\sigma(j)} \right) + \frac{1}{aj}\left( \frac{1}{a\sigma(j)} - \frac{1}{a\sigma(k)} \right)$$

$$S\sigma - S' = \frac{1}{ak}\left( \frac{a\sigma(j) - a\sigma(k)}{a\sigma(j) * a\sigma(k)} \right) + \frac{1}{aj}\left( \frac{a\sigma(k) - a\sigma(j)}{a\sigma(j) * a\sigma(k)} \right)$$

$$\ldots\ldots = \frac{(a\sigma(j) - a\sigma(k)) * (aj - ak)}{ak * aj * a\sigma(k) * a\sigma(j)} > 0$$

$$\Rightarrow a\sigma(j) - a\sigma(k) > 0 \Rightarrow a\sigma(j) > a\sigma(k) \Rightarrow \sigma = e.$$

$$\sigma = \begin{pmatrix} 1 \ldots 2 \ldots 3 \ldots\ldots n \\ n \ldots n-1 \ldots n-2 \ldots\ldots 1 \end{pmatrix}$$

13. Se dau numerele reale a1<a2< … <an.

Pentru ce permurare $\sigma \in S_n$ suma

$$S\sigma = \sum_{i=1}^{n} ai * a\sigma(i)$$

este maxima?

Fie τ =σo(k,j)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \ldots & k \ldots & j \ldots & n \\ \sigma_{(1)} & \sigma_{(2)} & \sigma_{(3)} \ldots & \sigma_{(k)} \ldots & \sigma_{(j)} \ldots & \sigma_{(n)} \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 \ldots & k \ldots & j \ldots & n \\ \sigma_{(1)} & \sigma_{(2)} & \sigma_{(3)} \ldots & \sigma_{(j)} \ldots & \sigma_{(k)} \ldots & \sigma_n \end{pmatrix}$$

$$S\tau = \sum_{i=1}^{n} ai * a\tau(i)$$

$$S\sigma - S\tau > 0$$

$$\Rightarrow S\sigma - S\tau = \sum_{i=1}^{n} ai * a\sigma(i) - \sum_{i=1}^{n} ai * a\tau(i)$$

$$\Rightarrow S\sigma - S\tau = \sum_{i=1}^{n} ai * a\sigma(i) + aj * a\sigma(j) + ak * a\sigma(k)$$

$$- \sum_{i=1}^{n} ai * a\tau(i) - aj * a\tau(j) - ak * a\tau(k)$$

$$\Rightarrow S\sigma - S\tau = aj * a\sigma(j) + ak * a\sigma(k) - aj * a\tau(j) - ak * a\tau(k)$$

$$\Rightarrow S\sigma - S\tau = aj * a\sigma(j) + ak * a\sigma(k) - aj * a\sigma(k) - ak * a\sigma(j)$$

$$\Rightarrow S\sigma - S\tau = aj[a\sigma(j) - a\sigma(k)] + ak[a\sigma(k) - a\sigma(j)]$$

$$\Rightarrow (aj - ak)[a\sigma(j) - a\sigma(k)] > 0$$

$$aj - ak > 0 \Rightarrow a\sigma(j) - a\sigma(k) > 0$$

$$\Rightarrow \sigma(j) > \sigma(k) \Rightarrow \sigma = e.$$