Trasarea graficului unei functii

In studiul variatiei unei functii si trasarea graficului se parcurg urmatoarele etape de determinare succesiva a unor elemente caracteristice ale functiei:

I. *Domeniul de definitie:*

a) Determinarea domeniului de definitie (in cazul expresiilor rationale numitorul trebuie sa fie diferit de zero; in cazul celor irationale cantitatea de sub radical trebuie sa fie cel putin zero)

b) Intersectia graficului cu axa *Ox*: *f(x)=0*

c) Intersectia graficului cu axa *Oy:f(0)=…*

d) Calculul limitelor:

$$\lim_{x->\infty} f(x) = ... \quad \text{si} \quad \lim_{x->-\infty} f(x) = ...$$

II. *Semnul functiei:*

a) Determinarea paritatii sau imparitatii functiei(daca functia este para,*f(x)=f(-x)*,atunci graficul este simetric fata de axa ordonatelor; daca functia este impara,*-f(x)=f(-x)*, atunci graficul este simetric fata de originea axelor).

b) Determinarea periodicitatii functiei si, in cazul functiilor periodice, a perioadei *T*.

c) Continuitatea functiei.

III. *Asimptote:*

a) orizontale;

b) oblice;

c) verticale.

IV. *Studiul primei derivate:*

a) Se determina multimea E` inclusa in domeniul de definitie, pe care functia *f* este derivabila si apoi se calculeaza *f `(x).*

b) Se rezolva ecuatia *f `(x)=0*, ale carei radacini sunt, eventual, puncte critice ale functiei.

c) Se calculeaza valoarile functiei pe radacinile derivatei I.

d) Determinarea semnului derivatei I, care da monotonia functiei.

V. *Studiul derivatei a doua:*

a) Se determina multimea E`` inclusa in E`, pe care functia *f `* este derivabila si apoi se calculeaza *f ``(x).*

b) Se rezolva ecuatia *f ``(x)=0*, iar radacinile pot fi puncte de inflexiune.

c) Se calculeaza valoarile functiei pe radacinile derivatei II.

d) Determinarea semnului derivateiei II, care ne da convexitatea sau concavitatea functiei.

VI. *Formarea tabloului de variatie a functiei f* – tablou in care se trec pentru sistematizare, rezultateleobtinute la punctele precedente:

x	
f `(x)	
f ``(x)	
f(x)	

VII. *Trasarea graficului functiei:*- conform rezultatelorsistematizate in tabloul de variatie – intr-un sistem de axe carteziene.

APLICATII:

1. Sa se studieze variatia functiilor si sa se reprezinte grafic:

a) $f(x) = \sqrt{|x^2 - 1|} - x$

I. a) $D = (-\infty, +\infty)$;

$$f(x) = \begin{cases} \sqrt{x^2 - 1} - x, & \text{daca } x \in (-\infty, -1) \cup (1, \infty) \\ \sqrt{1 - x^2} - x, & \text{daca } x \in [-1, 1] \end{cases}$$

b) $G_f \cap Ox : f(x) = 0 \Rightarrow \sqrt{|x^2 - 1|} - x = 0$

$$\Leftrightarrow |x^2 - 1| = x^2 \Leftrightarrow x^2 - 1 = \pm x^2 \Rightarrow 2x^2 = 1 \Rightarrow x = \frac{\sqrt{2}}{2} \quad \Rightarrow A(\frac{\sqrt{2}}{2}, 0)$$

c) $G_f \cap Oy : f(0) = 1 \quad \Rightarrow \quad B(0,1)$;

d) $\lim\limits_{n \to -\infty} f(x) = \lim\limits_{n \to -\infty} \sqrt{x^2 - 1} - x = +\infty$

$$\lim_{n \to \infty} f(x) = \lim_{n \to \infty} \sqrt{x^2 - 1} - x = \lim_{n \to \infty} \frac{x^2 - 1 - x^2}{\sqrt{x^2 - 1} + x} = 0$$

II. $f(-x) = \sqrt{x^2 - 1} + x$

III. asimptote orizontale : - spre $-\infty$ ____

- spre $+\infty$ $y = 0$

asimptota oblica spre $-\infty$: y = mx + n

$$m = \lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\sqrt{x^2-1}-x}{x} = \lim_{x\to-\infty}\frac{-x\sqrt{1-\frac{1}{x^2}}-x}{x} = \lim_{x\to-\infty}\frac{-x\left(\sqrt{1-\frac{1}{x^2}}+1\right)}{x} = -2$$

$$n = \lim_{x\to-\infty}\left(f(x)-mx\right) = \lim_{x\to-\infty}\left(\sqrt{x^2-1}-x+2x\right) = \lim_{x\to-\infty}\left(\sqrt{x^2-1}+x\right) = 0$$

$\Rightarrow y = -2x$ este asimptota oblica spre $-\infty$

asimptote verticale : ___

IV.
$$f`(x) = \begin{cases} \frac{x}{\sqrt{x^2-1}} - 1 \ , \ x \in (-\infty,-1)\cup(1,\infty) \\ -\frac{x}{\sqrt{1-x^2}} - 1 \ , \ x \in (-1,1) \end{cases}$$

$$\left.\begin{array}{l} f`_s(-1) = \lim\limits_{\substack{x\to-1\\x<-1}} f`(x) = \lim\limits_{\substack{x\to-1\\x<-1}} \frac{x}{\sqrt{x^2-1}} - 1 = \frac{-1}{0_+} - 1 = -\infty \\ f`_d(-1) = \lim\limits_{\substack{x\to-1\\x>-1}} f`(x) = \lim\limits_{\substack{x\to-1\\x>-1}} \frac{x}{\sqrt{1-x^2}} - 1 = \frac{-1}{0_-} = \infty \\ f(-1) = 1 \end{array}\right\} \Rightarrow M_1(-1,+1) \text{ - punct de intoarcere;}$$

$$\left.\begin{array}{l} f`_s(1) = \lim\limits_{\substack{x\to1\\x<1}} f`(x) = \lim\limits_{\substack{x\to1\\x<1}} \frac{x}{\sqrt{1-x^2}} - 1 = \frac{1}{0_-} - 1 = -\infty \\ f`_d(1) = \lim\limits_{\substack{x\to1\\x>1}} f`(x) = \lim\limits_{\substack{x\to1\\x>1}} \frac{x}{\sqrt{x^2-1}} - 1 = \frac{1}{0_+} = \infty \\ f(1) = -1 \end{array}\right\} \Rightarrow M_2(1,-1) \text{ - punct de intoarcere;}$$

$$f`(x) = 0 \Rightarrow -\frac{x}{\sqrt{1-x^2}} - 1 = 0 \Rightarrow x = -\frac{\sqrt{2}}{2}$$

$$f\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2} \Rightarrow C\left(-\frac{\sqrt{2}}{2}, \sqrt{2}\right)$$

x	-∞ -1 0 1 +∞
f `(x)	- - - -∞ | +∞ + 0 - - - - - - -∞ | +∞ + +
f (x)	+∞ 1 1 0 -1 0

➔ in –1 si 1 avem puncte de intoarcere.

b) $f(x)=\dfrac{|x|-3}{1+|x|}$

I. a) $D=(-\infty,\infty)$;

$$f(x)=\begin{cases}\dfrac{x-3}{x+1}, & x\in[0,\infty)\\ \dfrac{x+3}{x-1}, & x\in(-\infty,0)\end{cases}$$

b) $G_f\cap Ox:\quad f(x)=0\Rightarrow\dfrac{x-3}{x+1}=0\Rightarrow x-3=0\Rightarrow x=3\qquad A(3,0)$

$\Rightarrow\dfrac{x+3}{x-1}=0\Rightarrow x=-3\qquad A`(-3,0)$

c) $G_f\cap Ox:\quad f(0)=-3\quad\Rightarrow\quad B(0,-3)$

d) $\lim\limits_{x\to\pm\infty}f(x)=1$

II. $f(-x)=\dfrac{|x|-3}{1+|x|}=f(x)\Rightarrow$ functia este para, deci are graficul simetric fata de axa Oy

$\Rightarrow$ este suficient sa studiem functia pe restrictia sa $[0,\infty)$

III. asimptota orizontala: $y=1$

asimptota oblica: ___

asimptota verticala: ____

IV. $f`(x)=\left(\dfrac{x-3}{x+1}\right)`=\dfrac{x+1-x+3}{(x+1)^2}=\dfrac{4}{(x+1)^2}\Rightarrow f`(x)>0\Rightarrow f$ este strict crescatoare pe $[0,\infty)$

$\Rightarrow B(0.-3)$ este punct unghiular al functiei pe R (datorita simetriei sale fata de axa Oy)

V. $f``(x)=(f`(x))`=\left(\dfrac{4}{(x+1)^2}\right)`=-\dfrac{8}{(x+1)^4}\Rightarrow f``(x)<0\Rightarrow$ f este concava pe $(0,\infty)$

VI. Tabloul de variatie:

x	0			3						+∞
f `(x)	+	+	+	+	+	+	+	+	+	+
f``(x)	-	-	-	-	-	-	-	-	-	-
f(x)	-3			0						1

2. Se considera functia:

$$f : D \to R \qquad f(x) = \frac{2x^2 + 1}{x(x+k)}$$

unde *D* este domeniul maxim de definitie iar *k* partine lui *R*. Sa se traseze graficul functiei *f* stiind ca trce prin punctul (1,1).

<u>Demonstratie:</u>

Intrucat $M(1{,}1) \in G_f \Rightarrow f(1) = 1 \Leftrightarrow \frac{3}{1+k} = 1 \Rightarrow k = 2;$

$$\Rightarrow f(x) = \frac{2x^2 + 1}{x(x+2)}$$

$$\Rightarrow D = R - \{-2{,}0\}$$

I. a) $f : R - \{-2{,}0\} \to R$

b) $G_f \cap Ox : f(x) = 0 \Rightarrow 2x^2 + 1 = 0$

$\Delta < 0 \Rightarrow G_f$ nu intersecteaza axa Ox

c) $G_f \cap Oy : f(0) =$ _____

d) $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{2x^2+1}{x(x+2)} = 2$

II. $f(-x) = \frac{2x^2+1}{(-x)(-x+2)} = \frac{2x^2+1}{x(x-2)} \Rightarrow f$ este oarecare

III. asimptote orizontale : $y = 2$ spre $\pm\infty$

asimptote oblice : _____

asimptote verticale :

$$\left.\begin{array}{l} \lim\limits_{\substack{x\to-2\\x<-2}} f(x) = \lim\limits_{\substack{x\to-2\\x<-2}} \frac{2x^2+1}{x(x+2)} = \frac{9}{0_+} = \infty \\ \lim\limits_{\substack{x\to-2\\x>-2}} f(x) = \lim\limits_{\substack{x\to-2\\x>-2}} \frac{2x^2+1}{x(x+2)} = \frac{9}{0\ -} = -\infty \end{array}\right\} \Rightarrow x = -2 \text{ asimptota verticala}$$

$$\left.\begin{array}{l}\lim\limits_{\substack{x\to 0\\ x<0}} f(x)=\lim\limits_{\substack{x\to 0\\ x<0}}\dfrac{2x^2+1}{x(x+2)}=\dfrac{1}{0_-}=-\infty\\ \lim\limits_{\substack{x\to 0\\ x>0}} f(x)=\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{2x^2+1}{x(x+2)}=\dfrac{1}{0_+}=+\infty\end{array}\right\}\Rightarrow x=0 \text{ este asimptota verticala}$$

IV. $f`(x)=\left(\dfrac{2x^2+1}{x(x+2)}\right)^` =\dfrac{4x^2-2x-2}{x^2(x+2)^2}$

$f`(x)=0\Rightarrow 4x^2-2x-2=0\Rightarrow x_1=1;\ x_2=-\dfrac{1}{2}$

$f(1)=1;\quad f(-\dfrac{1}{2})=-2$

V.

x	-∞		-2			-1/2			0			1		∞
$f`(x)$	+	+	+ |	+	+	+	0	- - -	|	-	- - -	0	+	+ +
$f(x)$	2		+∞ |	-∞		-2		-∞ |	+∞			1		

3. Sa se reprezinte grafic functia:

$$f(x)=\frac{2(x-1)^2}{\sqrt{4x^2+2x+1}}$$

Demonstratie :

I. a) $f: R \to R$

b) $G_f \cap Ox : f(x)=0 \Rightarrow 2(x-1)^2=0 \Rightarrow x=1 \quad O(1,0)$

c) $G_f \cap Oy : f(0)=2 \quad A(0,2)$

d) $\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\frac{2(x-1)^2}{\sqrt{4x^2+2x+1}}=+\infty$

II. $f(-x)=\frac{2(-x-1)^2}{\sqrt{4x^2-2x+1}} \Rightarrow$ functia este oarecare

III. asimptote orizontale : ______

asimptote oblice : $y=mx+n$

$$m_1=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{2(x-1)^2}{x\sqrt{4x^2+2x+1}}=1$$

$$n_1=\lim_{x\to\infty}[f(x)-x]=\lim_{x\to\infty}\frac{2(x-1)^2-x\sqrt{4x^2+2x+1}}{\sqrt{4x^2+2x+1}}=$$

$$=\lim_{x\to\infty}\frac{4(x-1)^4-x^2(4x^2+2x+1)}{\sqrt{4x^2+2x+1}\left[2(x-1)^2+\sqrt{4x^2+2x+1}\right]}=$$

$$=\lim_{x\to\infty}\frac{-18x^3+23x^2-16x+4}{x^3\sqrt{4+2\frac{1}{x}+\frac{1}{x^2}}*\left[2\left(1-\frac{1}{x}\right)^2+\sqrt{4+2\frac{1}{x}+\frac{1}{x^2}}\right]}=-\frac{9}{4}$$

$\Rightarrow y=x-\frac{9}{4}$ este asimptota oblica pentru $x\to+\infty$

$$m_2=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{2(x-1)^2}{x\sqrt{4x^2+2x+1}}=-1$$

$$n_2=\lim_{x\to-\infty}[f(x)+x]=\lim_{x\to-\infty}\frac{2(x-1)^2+x\sqrt{4x^2+2x+1}}{\sqrt{4x^2+2x+1}}=$$

$$=\lim_{x\to\infty}\frac{4(x-1)^4+x^2(4x^2+2x+1)}{\sqrt{4x^2+2x+1}\left[2(x-1)^2-\sqrt{4x^2+2x+1}\right]}=$$

$$=\lim_{x\to\infty}\frac{-18x^3+23x^2-16x+4}{-x^3\sqrt{4+2\frac{1}{x}+\frac{1}{x^2}}*\left[2\left(1-\frac{1}{x}\right)^2+\sqrt{4+2\frac{1}{x}+\frac{1}{x^2}}\right]}=\frac{9}{4}$$

$\Rightarrow y = -x + \frac{9}{4}$ este asimptota oblica pentru $x \to -\infty$

IV. $f`(x) = \frac{2(x-1)(4x^2+7x+3)}{(4x^2+2x+1)^{3/2}}$

$$f`(x) = 0 \Rightarrow 2(x-1)(4x^2+7x+3) = 0 \Rightarrow x`_1 = 1;\ \ x`_2 = -1;\ \ x`_3 = -\frac{3}{4}$$

$$f(1) = 0;\ \ f(-1) = \frac{8\sqrt{3}}{3} = 4{,}6188;\ \ f(-\frac{3}{4}) = \frac{7\sqrt{7}}{4} = 4{,}63$$

V. $f``(x) = \frac{2(47x^2+46x+5)}{(4x^2+2x+1)^{5/2}}$

$$f``(x) = 0 \Rightarrow 47x^2+46x+5 = 0 \Rightarrow x = \frac{-23 \pm 7\sqrt{6}}{47} \Rightarrow$$

$\Rightarrow x``_1 = -0{,}854;\ \ \ x``_2 = -0{,}125$

$$f(x``_1) = 4{,}625;\ \ \ f(x``_2) = 2{,}805$$

V. Tabloul de variatie:

x	-∞ -1 -0,854 -3/4 -0,125 0 1 ∞
f `(x)	- - - 0 + + + + 0 - - - - - - 0 + +
f ``(x)	+ + + + 0 - - - - - 0 + + + + +
f(x)	+∞ 4,619 4,625 4,630 2,805 2 0 +∞

4. Sa se reprezinte grafic "*Serpentina lui Newton*" data prin functia:

$$f(x)=\frac{ax}{ax^2+1}\ ,\ a>0$$

Demonstratie :

I. a) $f:R\rightarrow R$

b) $G_f\cap Ox:\ f(x)=0\Rightarrow \frac{ax}{ax^2+1}=0\Rightarrow ax=0\Rightarrow x=0\ :\ O(0,0)$

c) $G_f\cap Oy:\ f(0)=0\Rightarrow O(0,0)$

d) $\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\frac{ax}{ax^2+1}=0$

II. $f(-x)=-\frac{ax}{ax^2+1}=-f(x)$

$\Rightarrow f$ este functie impara, deci G_f este simetric fata de origine.

III. asimptote orizontale : y = 0

asimptote oblice : ____

asimptote verticale : ____

IV. $f`(x)=\left(\frac{ax}{ax^2+1}\right)^{`}=\frac{a(ax^2+1)-2a^2x^2}{(ax^2+1)^2}=\frac{-a^2x^2+1}{(ax^2+1)^2}=a\frac{-ax^2+1}{(ax^2+1)^2}$

$f`(x)=0\Rightarrow a\frac{-ax^2+1}{(ax^2+1)^2}=0\Rightarrow -ax^2+1=0\Rightarrow x=\pm\frac{1}{\sqrt{a}}$

$f(\frac{1}{\sqrt{a}})=\frac{\sqrt{a}}{2};\quad f(-\frac{1}{\sqrt{a}})=-\frac{\sqrt{a}}{2}$

V. $f``(x)=\left(\frac{-a^2x^2+1}{(ax^2+1)^2}\right)^{`}=\frac{-2a^2x(ax^2+1)^2-4ax(-a^2x^2+1)(ax^2+1)}{(ax^2+1)^4}=\frac{2a^2x(a^2x^4-2ax^2-3)}{(ax^2+1)^4}=$

$=\frac{2a^2x(ax^2+1)(ax^2-3)}{(ax^2+1)^4}=\frac{2a^2x(ax^2-3)}{(ax^2+1)^3}$

$f``(x)=0\Rightarrow 2a^2x(ax^2-3)=0\Rightarrow x_1=0;\quad x_{2,3}=\pm\sqrt{\frac{3}{a}};\quad f\left(\sqrt{\frac{3}{a}}\right)=\frac{\sqrt{3a}}{4};\quad f\left(-\sqrt{\frac{3}{a}}\right)=-\frac{\sqrt{3a}}{4}$

x	-∞		- 3/a		- 1/ a		0		1/ a		3/a		+∞
f `(x)	-	-	-	-	0	+	+	+	0	-	-	-	-
f ``(x)	-	-	0	+	+	+	0	-	-	-	0	+	+
f(x)	0		- 3a /4		- a /2		0		a /2		3a /4		0

5. Sa se reprezinte grafic functia:

$$f(x)=\sqrt[3]{x^2}\pm\sqrt{a^2-x^2} \quad \text{(CARDIOIDA)}$$

I. a) $a^2-x^2\geq 0 \Leftrightarrow x^2\leq a^2 \Leftrightarrow x\leq |a| \Rightarrow x\in[-|a|,|a|]$

$f:[-|a|,|a|]\to R$

$$f(x)=\begin{cases}\sqrt[3]{x^2}+\sqrt{a^2-x^2}\\ \sqrt[3]{x^2}-\sqrt{a^2-x^2}\end{cases}$$

$f_1(x)=\sqrt[3]{x^2}+\sqrt{a^2-x^2}$

$f_2(x)=\sqrt[3]{x^2}-\sqrt{a^2-x^2}$

b) $G_f\cap Ox:\ f_1(x)=0;\ \sqrt[3]{x^2}+\sqrt{a^2-x^2}>0$

$\Rightarrow G_{f_1}$ nu intersecteaza axa Ox; se afla in intregime deasupra axei absciselor

$f_2(x)=0;\ \sqrt[3]{x^2}-\sqrt{a^2-x^2}=0 \Leftrightarrow \sqrt[3]{x^2}=\sqrt{a^2-x^2}\uparrow^6 \Leftrightarrow$

$\Leftrightarrow x^4=(a^2-x^2)^3 \Leftrightarrow x^4=a^6-2a^4x^2+a^2x^4-a^4x^2+2a^2x^4-x^6$

$\Leftrightarrow x^6-x^4(3a^2-1)+3a^4x^2-a^6=0$

Fie $g(x)=x^6-x^4(3a^2-1)+3a^4x^2-a^6$

$$\left.\begin{array}{l}g(0)=-a^6\\ g(\pm a)=a^4\end{array}\right\}\Rightarrow g(0)g(\pm a)=(-a^6)a^4<0 \xrightarrow{\text{lema intersectiei}} (\exists)\text{cel putin un } x \text{ astfe incat } g(x)=0$$

$\Rightarrow x_1\in(0,|a|);\ x_2\in(-|a|,0)$

c) $G_f\cap Oy:\ f_1(0)=|a|;\quad f_2(0)=-|a|$

$f(\pm a)=f_1(\pm a)=f_2(\pm a)=\sqrt[3]{a^2}$

d) $\lim\limits_{\substack{x\to|a|\\ x\leq|a|}} f(x)=\sqrt[3]{a^2}$

$\lim\limits_{\substack{x\to-|a|\\ x\geq-|a|}} f(x)=\sqrt[3]{a^2}$

II. $f(-x)=f(x)\Rightarrow$ functia este para, deci graficul asociat este simetric fata de axa ordonatelor

III. asimptote orizontale : ___

asimptote oblice : ___

asimptote verticale : ___

IV. $f`(x) = \frac{2}{3}\frac{1}{\sqrt[3]{x}} \mp \frac{x}{\sqrt{a^2 - x^2}}$

$f_1`(x) = \frac{2}{3}\frac{1}{\sqrt[3]{x}} - \frac{x}{\sqrt{a^2 - x^2}}$

$f_2`(x) = \frac{2}{3}\frac{1}{\sqrt[3]{x}} + \frac{x}{\sqrt{a^2 - x^2}}$

$f`(x) = 0 \Leftrightarrow f_1`(x) = 0; \ f_2`(x) = 0$

- pentru $x \in (0,|a|)$ $f_2`(x) > 0 \Rightarrow f_2$ este strict crescatoare pe $(0,|a|)$
- pentru $x \in (-|a|,0)$ $f_1`(x) < 0 \Rightarrow f_2$ este strict descrescatoare pe$(-|a|,0)$

$f_1`(x) = 0 \Leftrightarrow 2\sqrt{a^2 - x^2} - 3x\sqrt[3]{x} = 0 \Leftrightarrow 2\sqrt{a^2 - x^2} = 3x\sqrt[3]{x} \uparrow^6$

$\Rightarrow 729x^8 + 64x^6 - 192a^2x^4 + 192a^4x^2 - 64a^6 = 0$

fie $h(x) = 729x^8 + 64x^6 - 192a^2x^4 + 192a^4x^2 - 64a^6$

$\left.\begin{array}{l} h(0) = -64a^6 \\ h(\pm a) = 729a^8 \end{array}\right\} \Rightarrow h(0)h(\pm a) = (-64a^6)729a^8 < 0 \xrightarrow{\text{lema intersectiei}} (\exists)x`_1 \in (0,|a|)$ si $x`_2 \in (-|a|,0)$

astfel incat $f_1`(x`) = 0 \Rightarrow x`_1$ si $x`_2$ sunt puncte de extrem

$\lim_{\substack{x\to 0\\ x>0}} f`(x) = +\infty$ si $\lim_{\substack{x\to 0\\ x<0}} f`(x) = -\infty \Rightarrow$ punctele de coordonate: $(0, f(0))$, adica

$(0,|a|)$ si $(-|a|,0)$ sunt puncte unghiulare

Deoarece $\lim_{\substack{x\to |a|\\ x<|a|}} f`(x) = -\infty$ si $\lim_{\substack{x\to -|a|\\ x>-|a|}} f`(x) = +\infty$ atunci G_{f_1} este tangent la dreptele de ecuatie: $x = \pm|a|$

V. $f``(x) = -\frac{2}{9}\frac{1}{\sqrt[3]{x^4}} \mp \frac{a^2}{(a^2 - x^2)^{3/2}}$

$f_1``(x) = -\frac{2}{9}\frac{1}{\sqrt[3]{x^4}} - \frac{a^2}{(a^2 - x^2)^{3/2}}$

$f_2``(x) = -\frac{2}{9}\frac{1}{\sqrt[3]{x^4}} + \frac{a^2}{(a^2 - x^2)^{3/2}}$

– pentru $x \in [-|a|,|a|]$ $f_1``(x) < 0 \Rightarrow$ ramura f_1 este concava.

– pentru $f_2``(x) = 0 \Rightarrow$

$$-\frac{2}{9}\,\frac{1}{\sqrt[3]{x^4}}+\frac{a^2}{(a^2-x^2)^{3/2}}=0 \Leftrightarrow \frac{2}{9}\,\frac{1}{\sqrt[3]{x^4}}=\frac{a^2}{(a^2-x^2)^{3/2}} \quad \uparrow^6$$

$$\Rightarrow 64(a^2-x^2)^9-9^6a^{12}x^8=0$$

Fie $j(x)=64(a^2-x^2)^9-9^6a^{12}x^8$

$$\left.\begin{matrix} j(0)=64a^{18} \\ j(\pm a)=-9^6a^{20} \end{matrix}\right\}\Rightarrow j(0)\,j(\pm a)=(-9^6a^{20})64a^{18}<0 \to (\exists)x``_2\in(-|a|,0) \text{ si } x``_1\in(0,|a|)$$

astfel incat $f_2``(x)=0 \Rightarrow x``_1$ si $x``_2$ puncte de inflexiune

VI. Tabloul de variatie al functiei se face separat pentru cele doua ramuri:

x	-|a| x`2     0     x`1 |a|
f `1(x)	+ + + 0 - - | + + 0 - - - -
f ``1(x)	- - - - - - - - - - - - -
f 1(x)	a |a| a

x	-|a| x``2     0     x``1 |a|
f `2(x)	- - - - | + + + +
f ``2(x)	+ + 0 - - | - - 0 + +
f2(x)	a -|a| a