INTEGRAREA UNOR EXPRESII IRATIONALE

1) $\int R(x;\sqrt[n]{ax+b})dx$

Notatie $ax+b=t^n$

2) $\int R(x;\int R(x;\sqrt[n1]{ax+b})dx;.....;\sqrt[nk]{ax+b})dx$

Notatie $ax+b=t^m$ $m=c.m.m.m.c(n_1,n_2,....,n_k)$

3) $\int (x;\sqrt[n]{\frac{ax+b}{cx+d}})dx$

Notatie $\frac{ax+b}{cx+d}=t^n$

4) $\int (x;\sqrt[n1]{\frac{ax+b}{cx+d}};.......;\sqrt[nk]{\frac{ax+b}{cx+d}})dx$

Notatie $\frac{ax+b}{cx+d}=t^m$ $m=c.m.m.m.c(n_1,n_2,....,n_k)$

5) $\int R(x;\sqrt{ax^2+bx+c})$

i) $a>0$

$\sqrt{ax^2+bx+c}=t+x\sqrt{a}$ sau$(t-x\sqrt{a})$

ii) $c>0$

$\sqrt{ax^2+bx+c}=tx+\sqrt{c}$ sau$(tx-\sqrt{c})$

iii) $\Delta>0$ $(\alpha,\beta$ radacini)

$\sqrt{ax^2+bx+c}=t(x-\alpha)$ sau $t(x-\beta)$

APLICATII

Sa se calculeze primitivele urmatoarelor functii :

1) $I=\int\sqrt{\frac{1-x}{1+x}}dx$ $x\in(-1,1)$

$$\frac{1-x}{1+x}=t^2$$

$$1-x=t^2+t^2x$$

$$x(t^2+1)=1-t^2$$

$$x = \frac{1 - t^2}{1 + t^2}$$

$$dx = \frac{-2t(1 + t^2) + 2t(1 - t^2)}{(1 + t^2)^2} dt$$

$$dx = \frac{-2t - 2t^3 + 2t - 2t^3}{(1 + t^2)^2} dt$$

$$dx = \frac{-4t^3}{(1 + t^2)^2} dt$$

$$I' = -4\int \frac{t^4}{(1 + t^2)^2} dt = -4[\int dt - \int \frac{2t^2 + 1}{(1 + t^2)^2} dt]$$

$$I' = -4[t - 2\int \frac{t^2}{(1 + t^2)^2} dt + \int \frac{1}{(1 + t^2)^2} dt]$$

$$I' = -4[t - 2\int \frac{t^2 + 1 - 1}{(1 + t^2)^2} dt + \int \frac{1}{(1 + t^2)^2} dt]$$

$$I' = -4[t - 2\int \frac{1}{(1 + t^2)} dt + 3\int \frac{1}{(1 + t^2)^2} dt]$$

$$I_1 = \int \frac{1}{(1 + t^2)^2} dt = \int \frac{1 + t^2 - t^2}{(1 + t^2)^2} dt$$

$$I_1 = \int \frac{1}{1+t^2}\,dt - \int t\,\frac{t}{(1+t^2)^2}\,dt$$

$$f(t) = t \qquad \mathrm{g}'(\mathrm{t}) = \frac{t}{(1+t^2)^2}$$

$$f'(\mathrm{t}) = 1 \qquad \mathrm{g}(\mathrm{t}) = -\frac{1}{2(1+t^2)}$$

$$I_1 = \int \frac{1}{1+t^2}\,dt + \frac{\mathrm{t}}{2(1+t^2)} + \frac{1}{2}\int \frac{1}{1+t^2}\,dt$$

$$I_1 = \frac{3}{2}\,\mathrm{arctg\ t} + \frac{\mathrm{t}}{2(1+t^2)}$$

$$I' = -4t - \frac{7}{2}\,\mathrm{arctg\ t} - \frac{6\mathrm{t}}{(1+t^2)}$$

$$\mathrm{I} = -4\sqrt{\frac{1-\mathrm{x}}{1+\mathrm{x}}} - \frac{7}{2}\,\mathrm{arctg}\sqrt{\frac{1-\mathrm{x}}{1+\mathrm{x}}} - \frac{6\sqrt{\frac{1-\mathrm{x}}{1+\mathrm{x}}}}{\frac{2}{1+\mathrm{x}}}$$

$$\mathrm{I} = -4\sqrt{\frac{1-\mathrm{x}}{1+\mathrm{x}}} - \frac{7}{2}\,\mathrm{arctg}\sqrt{\frac{1-\mathrm{x}}{1+\mathrm{x}}} - 3(1+x)\sqrt{\frac{1-\mathrm{x}}{1+\mathrm{x}}}$$

$$2)\ \mathrm{I} = \int \frac{\mathrm{x} - 1}{\sqrt[3]{x} + 1} dx \qquad \mathrm{x} \in \mathrm{R} \setminus \{-1\}$$

$$\mathrm{x} = t^3$$

$$dx = 3t^2 dt$$

$$\mathrm{I} = \int \frac{\mathrm{x}}{\sqrt[3]{x} + 1} dx - \int \frac{1}{\sqrt[3]{x} + 1} dx$$

$$I' = 3\int \frac{t^5}{t + 1} dt - 3\int \frac{t^2}{t + 1} dt$$

$$\frac{t^5}{t + 1} = t^4 - t^3 + t^2 - t + 1 - \frac{1}{t + 1}$$

$$I' = 3\int (t^4 - t^3 + t^2 - t + 1 - \frac{1}{t + 1}) dt$$

$$- 3\int (\mathrm{t} - 1 + \frac{1}{1 + \mathrm{t}}) dt$$

$$I' = 3[\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ln(t + 1)] -$$

$$- 3[\frac{t^2}{2} - \mathrm{t} + \ln(\mathrm{t} + 1)]$$

$$I = 3\left[\frac{x\sqrt[3]{x^2}}{5} - \frac{x\sqrt[3]{x}}{4} + \frac{x}{3} - \frac{\sqrt[3]{x^2}}{2} + \sqrt[3]{x} - \ln(\sqrt[3]{x}+1)\right]$$

$$-\,3\left[\frac{\sqrt[3]{x^2}}{2} - \sqrt[3]{x} + \ln(\sqrt[3]{x}+1)\right]$$

$$I = 3\left(\frac{x\sqrt[3]{x^2}}{5} - \frac{x\sqrt[3]{x}}{4} + \frac{x}{3}\right) + C$$

3) $\mathrm{I} = \int \frac{\mathrm{dx}}{(1+\mathrm{x})\sqrt{1+\mathrm{x}+x^2}}$ $\quad\quad$ $\mathrm{x} \in \mathrm{R} \setminus \{-1\}$

$$\mathrm{a} > 0 \Rightarrow \sqrt{1+\mathrm{x}+x^2} = t + x \,|^2$$

$$1+\mathrm{x}+x^2 = t^2 + 2tx + x^2$$

$$x(1-2t) = t^2 - 1$$

$$x = \frac{t^2-1}{1-2t}$$

$$dx = \frac{2t(1-2t) + 2(t^2-1)}{(1-2t)^2}\,dt$$

$$dx = \frac{2t - 4t^2 + 2t^2 - 2}{(1-2t)^2}\,dt$$

$$dx = \frac{-2t^2 + 2t - 2}{(1-2t)^2}\,dt$$

$$1+x=1+\frac{t^2-1}{1-2t}=\frac{1-2t+t^2-1}{1-2t}=\frac{t^2-2t}{1-2t}$$

$$\sqrt{1+x+x^2}=t+x=t+\frac{t^2-1}{1-2t}=\frac{t-2t^2+t^2-1}{1-2t}=$$

$$=\frac{-t^2+t-1}{1-2t}$$

$$\frac{1}{(1+x)\sqrt{1+x+x^2}}=\frac{1}{\frac{t^2-2t}{1-2t}\cdot\frac{-t^2+t-1}{1-2t}}=$$

$$=\frac{(1-2t)^2}{t(t-2)(-t^2+t-1)}$$

$$I'=\int\frac{(1-2t)^2}{t(t-2)(-t^2+t-1)}\frac{2(-t^2+t-1)}{(1-2t)^2}dt$$

$$I'=2\int\frac{1}{t(t-1)}dt$$

$$\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}=\frac{(A+B)t-A}{t(t-1)}$$

$$\begin{cases} A+B=0 \\ -A=1 \Rightarrow A=-1 \end{cases}$$

$$B=1$$

$$I^{'} = 2[-\int \frac{1}{t} dt + \int \frac{1}{t-1} dt]$$

$$I^{'} = 2[-\ln t + \ln(t-1)]$$

$$I^{'} = 2[-\ln(\sqrt{1+x+x^2} - x) + \ln(\sqrt{1+x+x^2} - x - 1)]$$

$$4) I = \int \frac{x^2+x+1}{x\sqrt{x^2-x+1}} dx$$

$$c > 0$$

$$\sqrt{x^2-x+1} = tx+1 \; |^2$$

$$x^2 - x + 1 = t^2x^2 + 2tx + 1$$

$$x^2 - x = t^2x^2 + 2tx \quad |: \mathrm{x}$$

$$\mathrm{x} - 1 = t^2x + 2t$$

$$x(1-t^2) = 2t - 1$$

$$x = \frac{2t-1}{(1-t^2)}$$

$$dx = \frac{2(1-t^2)-(2t-1)(-2t)}{(1-t^2)^2}dt$$

$$dx = \frac{2-2t^2+4t^2-2t}{(1-t^2)^2}dt$$

$$dx = \frac{2t^2-2t+2}{(1-t^2)^2}dt$$

$$\sqrt{x^2-x+1} = t\frac{2t-1}{1-t^2}+1 = \frac{2t^2-t+1-t^2}{1-t^2} =$$

$$= \frac{t^2-t+1}{1-t^2}$$

$$I = \int \frac{x^2}{x\sqrt{x^2-x+1}}dx + \int \frac{x}{x\sqrt{x^2-x+1}}dx +$$

$$+\int \frac{dx}{x\sqrt{x^2-x+1}}$$

$$I = \int \frac{x}{\sqrt{x^2-x+1}}dx + \int \frac{dx}{\sqrt{x^2-x+1}} + \int \frac{dx}{x\sqrt{x^2-x+1}}$$

$$\frac{x}{\sqrt{x^2-x+1}} = \frac{(2t-1)}{(1-t^2)}\frac{(1-t^2)}{(t^2-t+1)} = \frac{2t-1}{t^2-t+1}$$

$$\frac{1}{\sqrt{x^2-x+1}}=\frac{1-t^2}{t^2-t+1}$$

$$\frac{1}{x\sqrt{x^2-x+1}}=\frac{(1-t^2)}{(2t-1)}\frac{(1-t^2)}{(t^2-t+1)}=\frac{(1-t^2)^2}{(2t-1)(t^2-t+1)}$$

$$I' = \int \frac{2t-1}{t^2-t+1}.\frac{2(t^2-t+1)}{1-t^2}dt+\int \frac{1-t^2}{t^2-t+1}.\frac{2(t^2-t+1)}{(1-t^2)^2}dt$$

$$+\int \frac{(1-t^2)^2}{(2t-1)(t^2-t+1)}.\frac{2(t^2-t+1)}{(1-t^2)^2}dt$$

$$I' = 2\int \frac{2t-1}{(1-t^2)^2}dt+2\int \frac{dt}{1-t^2}+2\int \frac{dt}{2t-1}$$

$$\frac{2t-1}{(1-t)^2(1+t)^2}=\frac{A}{1-t}+\frac{B}{(1-t)^2}+\frac{C}{1+t}+\frac{D}{(1+t)^2}$$

$$2t-1=A(1-t)(1+t)^2+B(1+t)^2+C(1+t)(1-t)^2+$$

$$+\mathrm{D}(1-t)^2$$

$$2t-1=A(-t^3-t^2+t+1)+B(t^2+2t+1)+$$

$$+\mathrm{C}(t^3-t^2-\mathrm{t}+1)+\mathrm{D}(1-2\mathrm{t}+t^2)$$

$$\begin{cases} -A+C=0 \Rightarrow C=A \\ -A+B-C+D=0 \\ A+2B-C-2D=2 \\ A+B+C+D=-1 \end{cases}$$

$$\begin{cases} -2A+B+D=0 \\ \quad 2B-2D=2 \quad \Rightarrow \quad B=1+D \\ 2A+B+D=-1 \end{cases}$$

$$\begin{cases} -2A+2D=-1 \\ 2A+2D=-2 \end{cases}$$

$4D=-3$

$D=-3/4$; $A=-1/4$; $C=-1/4$; $B=-1/4$

$$\frac{1}{1-t^2}=\frac{A}{1-t}+\frac{B}{1+t}$$

$$1=A(1+t)+B(1-t)$$

$$\begin{cases} A-B=0 \\ A+B=1 \end{cases}$$

$$2A=1$$

$$A=\frac{1}{2}\quad\Rightarrow \mathrm{B}=\frac{1}{2}$$

$$I^{'}=2\left[\frac{1}{4}\int\frac{-1}{1-t}dt-\frac{1}{4}\int\frac{-1}{(1-t)^{2}}dt-\frac{1}{4}\int\frac{dt}{1+t}-\frac{3}{4}\int\frac{dt}{(1+t)^{2}}\right]$$

$$+\int\frac{\mathrm{dt}}{1-\mathrm{t}}+\int\frac{\mathrm{dt}}{1+\mathrm{t}}+2\int\frac{\mathrm{dt}}{2\mathrm{t}-1}$$

$$I^{'}=\frac{1}{2}\ln(1-t)+\frac{1}{2(1-t)}-\frac{1}{2}\ln(1+t)+\frac{3}{2(1+t)}-\ln(1-t)+$$

$$+\ln(1+\mathrm{t})+\ln(2\mathrm{t}-1)$$

$$I^{'}=-\frac{1}{2}\ln(1-t)+\frac{1}{2}\ln(1+t)+\ln(2t-1)+\frac{1}{2(1-t)}+\frac{3}{2(1+t)}$$

$$t=\frac{\sqrt{x^{2}-x+1}-1}{x}$$

$$I^{'}=-\frac{1}{2}[\ln\frac{x-\sqrt{x^{2}-x+1}-1}{x}-\ln\frac{x+\sqrt{x^{2}-x+1}-1}{x}$$

$$-\ln\frac{2\sqrt{x^{2}-x+1}-1-x}{x}+\frac{x}{2(x-\sqrt{x^{2}-x+1}-1)}$$

$$+\frac{3x}{2(x+\sqrt{x^{2}-x+1}-1)}]$$

$$5)\, I = \int \frac{dx}{(x-1)\sqrt{x^2-3x+2}} \quad , x \in (-\infty,1)\ \text{Y}(2,\infty)$$

$$\Delta = 1 > 0$$

$$x_1 = 1; x_2 = 2$$

$$\sqrt{x^2-3x+2} = t(x-1) \ \ |^2$$

$$x^2-3x+2 = t^2(x-1)^2$$

$$(x-1)(x-2) = t^2(x-1)^2 \quad |:(\mathrm{x}-1)$$

$$\mathrm{x} - 2 = t^2x - t^2$$

$$x(1-t^2) = 2-t^2$$

$$x = \frac{2-t^2}{1-t^2}$$

$$dx = \frac{-2t(1-t^2)+2t(2-t^2)}{(1-t^2)^2}dt$$

$$dx = \frac{-2t+2t^3+4t-2t^3}{(1-t^2)^2}dt$$

$$dx = \frac{2t}{(1-t^2)^2}dt$$

$$x-1=\frac{2-t^2}{1-t^2}-1=\frac{1}{1-t^2}$$

$$\sqrt{x^2-3x+2}=\frac{t}{1-t^2}$$

$$I^{'}=\int\frac{(1-t^2)^2}{t}\frac{2t}{(1-t^2)^2}dt=2\int dt=2t+c$$

$$t=\sqrt{\frac{x-2}{x-1}}$$

$$I=2\sqrt{\frac{x-2}{x-1}}+c$$

$$6)I=\int\frac{\ln x}{x\sqrt{1-4\ln x-\ln^2 x}}dx\,;\mathrm{x}\in\left(e^{-2-\sqrt{6}},e^{-2+\sqrt{6}}\right)\setminus\{0\}$$

$$\ln x=t$$

$$x=e^t\ \Rightarrow \mathrm{dx}=e^t dt$$

$$I^{'}=\int\frac{t}{e^t\sqrt{1-4\mathrm{t}-\mathrm{t}^2}}e^t\mathrm{dt}=\int\frac{t}{\sqrt{1-4\mathrm{t}-\mathrm{t}^2}}\mathrm{dt}$$

$$\mathrm{c}>0\Rightarrow\sqrt{1-4\mathrm{t}-\mathrm{t}^2}=zt+1\ |^2$$

$$1-4\mathrm{t}-\mathrm{t}^2=z^2t^2+2zt+1$$

$$-4\mathrm{t}-\mathrm{t}^2=z^2t^2+2zt\ \ |:\mathrm{t}$$

$$-4-t=z^2t+2z$$

$$-4-2z=t(z^2+1)$$

$$t=\frac{-4-2z}{z^2+1}$$

$$dt=\frac{-2(z^2+1)-2z(-4-2z)}{(z^2+1)^2}dz$$

$$dt=\frac{-2z^2-2+8z+4z^2}{(z^2+1)^2}dz$$

$$dt=\frac{2z^2+8z-2}{(z^2+1)^2}dz$$

$$\sqrt{1-4t-t^2}=z\frac{-4-2z}{z^2+1}+1=\frac{-4z-2z^2+z^2+1}{z^2+1}=$$

$$=\frac{-z^2-4z+1}{z^2+1}$$

$$I''=\int\frac{-4-2z}{z^2+1}\cdot\frac{z^2+1}{-z^2-4z+1}\cdot\frac{2(-z^2-4z+1)}{(z^2+1)^2}dz$$

$$I''=\int\frac{z+2}{(z^2+1)^2}dz=\int\frac{z}{(z^2+1)^2}dz+2\int\frac{dz}{(z^2+1)^2}$$

$$I_1 = \int \frac{1}{(z^2+1)^2} dz = \int \frac{z^2+1-z^2}{(z^2+1)^2} dz =$$

$$= \int \frac{1}{z^2+1} dz - \int \frac{z^2}{(z^2+1)^2} dz$$

$$f(z) = \mathrm{z} \qquad \mathrm{g}'(z) = \frac{z}{(z^2+1)^2}$$

$$f'(z) = 1 \qquad \mathrm{g(z)} = \frac{-1}{2(z^2+1)}$$

$$I_1 = arctg(z) + \frac{z}{2(z^2+1)} - \frac{1}{2} arctg(z)$$

$$I_1 = \frac{1}{2} arctg(z) + \frac{z}{2(z^2+1)} + c1$$

$$I'' = \frac{-1}{2(z^2+1)} + arctg(z) + \frac{z}{(z^2+1)} + c2$$

$$\mathrm{z} = \frac{\sqrt{1-4t-t^2}-1}{t}$$

$$I' = \frac{-t^2}{4(1-2t-\sqrt{1-4t-t^2})} + arctg\left(\frac{\sqrt{1-4t-t^2}-1}{t}\right) +$$

$$+\frac{2(1-2t-\sqrt{1-4t-t^2})(\sqrt{1-4t-t^2}-1)}{t^3} + c3$$

$$I = \frac{-\ln^2 x}{4(1-2\ln x-\sqrt{1-4\ln x-\ln^2 x})}$$

$$+arctg\left(\frac{\sqrt{1-4\ln x-\ln^2 x}-1}{\ln x}\right) +$$

$$+\frac{2(1-2\ln x-\sqrt{1-4\ln x-\ln^2 x})(\sqrt{1-4t-\ln^2 x}-1)}{\ln^3 x} + c$$