Integrarea diferentialelor binome
Substitutile lui Cebisev

Calculul primitivelor de forma:

$\int x^m (ax^n + b)^p \cdot dx$ unde $a, b \in R$ si $m, n, p \in Q$.

Daca p sau $\frac{m+1}{n}$ sau $p + \frac{m+1}{n} \in Z$, atunci calculul primitivelor date se reduce la calculul primitivei dintr-o functie rationala .

Intr-adevar , cu substitutia $x = t^{\frac{1}{n}}$, avem $dx = \frac{1}{n} t^{\frac{1}{n-1}} \cdot dt$, deci

$$F = \int x^m (ax^n + b)^p \cdot dx = \frac{1}{n} \int t^{\frac{m+1}{n}-1} (at + b)^p \cdot dt = \frac{1}{n} \int t^q (at + b)^p \cdot dt \text{ .}$$

Cazul 1.

$p \in Z$

Sa punem $q = \frac{r}{s}$ unde $r, s \in Z, s \neq 0$.Atunci substitutia

$t^{\frac{1}{s}} = y \Leftrightarrow t = y^s$

ne da $dt = sy^{s-1} dy$,deci

$$F = \frac{s}{n} \int y^{r+s-1} (ay^s + b)^p \cdot dy = \int R(y) \cdot dy$$

unde R este functie rationala deoarece $r, s, p \in Z$.

Cazul 2.

$$\frac{m+1}{n} \in Z \Leftrightarrow q = \frac{m+1}{n} - 1 \in Z$$

Sa punem $p = \frac{r}{s}$, unde $r, s \in Z, s \neq 0$.Atunci substitutia $(at+b)^{\frac{1}{s}} = y \Leftrightarrow t = \frac{y^s - b}{a}$,ne da $dt = \frac{s}{a} y^{s-1} \cdot dy$,deci

$$F = \frac{s}{na} \int \left(\frac{y^s - b}{a} \right)^q \cdot y^{r+s-1} \cdot dy = \int R(y) \cdot dy$$

unde R este functie rationala deoarece $r, s, p \in Z$.

Cazul 3.

$$p + \frac{m+1}{n} \in Z \Leftrightarrow p + q = p + \frac{m+1}{n} - 1 \in Z$$

Evident avem $F = \frac{1}{n} \int t^{p+q} \left(a + bt^{-1}\right)^p \cdot dt$

Sa punem $p = \frac{r}{s}$, unde $r, z \in Z, s \neq 0$. Atunci substitutia $\left(a + bt^{-1}\right)^{\frac{1}{s}} = y \Leftrightarrow t = \frac{b}{y^s - a}$, ne da $dt = -\frac{sb + y^{s-1}}{\left(y^s - a\right)^2} \cdot dy$,deci

$$F = -\int \left(\frac{b}{y^s - a} \right)^{p+q} \cdot y^r \frac{sby^{s-1}}{\left(y^s - a\right)^2} \cdot dy = \int R(y) \cdot dy$$

unde R este functie rationala deoarece $s, r, p+q \in Z$.

Concluzie.

Prin urmare substitutile urmatoare :

1. $\left(x^n\right)^{\frac{1}{s}} = y$, daca $p \in Z$, unde $\frac{m+1}{n} = \frac{r}{s}$;
2. $\left(ax^n + b\right)^{\frac{1}{s}} = y$, daca $\frac{m+1}{n} \in Z$, unde $p = \frac{r}{s}$;

3. $\left(a+bx^{-n}\right)^{\frac{1}{s}}=y$, daca $\frac{m+1}{n}+p\in Z$, unde $p=\frac{r}{s}$,

reduc calculul primitivei $\int x^m\left(ax^n+b\right)^p\cdot dx$ la calculul primitivei dintr-o functie rationala .

Observatie.

Cebisev a aratat ca daca p , $\frac{m+1}{n}$ si $p+\frac{m+1}{n}\in Z$,atunci primitiva data nu se poate reduce la primitiva dintr-o functie rationala . Calculul primitivei nu poate fi facut atunci prin mijloace elementare .

Exemplul 1.

Sa se calculeze primitiva $F=\int x^{\frac{5}{4}}\left(1+x^{\frac{3}{5}}\right)^{-2}\cdot dx$.

Avem $p=-3\in Z$, deci suntem in cazul 1.

Cum $\frac{m+1}{n}=\frac{15}{4}$ facem substitutia

$\left(x^{\frac{3}{5}}\right)^{\frac{1}{4}}=t\Leftrightarrow x=t^{\frac{20}{3}}$, deci $dx=\frac{20}{3}t^{\frac{17}{3}}\cdot dt$ si deci

$$F=\int t^{\frac{20}{3}\cdot\frac{5}{4}}\left(1+t^{\frac{20}{3}\cdot\frac{3}{5}}\right)^{-2}\cdot\frac{20}{3}t^{\frac{17}{3}}\cdot dt=\frac{20}{3}\int\frac{t^{14}\cdot dt}{\left(1+t^4\right)^2}$$

Exemplul 2.

Sa se calculeze primitiva $F=\int x^3\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}\cdot dx$

Avem $m=3,n=\frac{3}{2},deci\,\frac{m+1}{n}\in Z$ si deci suntem in cazul 2.

Facem substitutia $\left(1-x^{\frac{2}{3}}\right)^{\frac{1}{2}}=t$ .Atunci $x^{\frac{2}{3}}=1-t^2, \frac{2}{3}x^{-\frac{1}{3}}\cdot dx=-2t\cdot dt$, de unde obtinem :

$$F=\int -3x^3t^3x^{\frac{1}{3}}t\cdot dt=-3\int x^{\frac{4}{3}}t^4\cdot dt=-3\int t^4\left(1-t^2\right)^4\cdot dt=-3\left(\frac{t^5}{5}-\frac{2t^7}{7}+\frac{t^9}{9}\right)$$

Exemplul 3.

Sa se calculeze primitiva $F=\int x^{-\frac{1}{2}}\left(1-x^{-\frac{4}{3}}\right)^{-\frac{5}{8}}\cdot dx$

Avem $m=-\frac{1}{2}$, $n=-\frac{4}{3}$ si $p=-\frac{5}{2}$,deci $\frac{m+1}{n}+p=-1\in Z$ si deci

suntem in cazul 3. Facem substitutia $\left(x^{\frac{4}{3}}-1\right)^{\frac{1}{8}}=t$. Atunci

$dx=6x^{-\frac{1}{3}}t^7\cdot dt$, de unde obtinem :

$$F=\int x^{\frac{1}{3}}t^{-5}6x^{-\frac{1}{3}}t^7\cdot dt=\int t^2\cdot dt=2t^3=2\left(x^{\frac{4}{3}}-1\right)^{\frac{3}{8}}$$

Exemplul 4.

Sa se calculeze primitiva $F=\int \frac{dx}{x^2\cdot\sqrt{x^2-1}}$

Avem functia F= $x^2\left(x^2-1\right)^{-\frac{1}{2}}$

unde $\frac{m+1}{n}=-\frac{1}{2}$

$$\frac{m+1}{n}+p=-\frac{1}{2}-\frac{1}{2}=-1\in Z$$

Facem substitutia

$ax^n+b=x^nt^\alpha \quad \Rightarrow x^2-1=x^2t^2$

$1-\frac{1}{x^2}=t^2 \quad \Rightarrow \frac{1}{x^2}=1-t^2 \quad \Rightarrow x^2=\frac{1}{1-t^2} \quad \Rightarrow x=\sqrt{\frac{1}{1-t^2}}$

$\Rightarrow -\frac{2}{x^3}dx = -2t \cdot dt \quad \Rightarrow \frac{dx}{x^3} = t \cdot dt$ si obtinem :

$$F = \int \frac{x^3 t \cdot dt}{x^2 \cdot \sqrt{x^2 - 1}} = \int \frac{xt \cdot dt}{\left(x^2 t^2\right)^{\frac{1}{2}}} = \int \frac{xt}{xt} \cdot dt = \int dt = t + c = \frac{\sqrt{x^2 - 1}}{x} + c$$

Exemplul 5.

Sa se calculeze primitiva $F = \int \frac{x \cdot dx}{\sqrt[3]{x^2 + 2}} = \int x^1 \left(x^2 + 2\right)^{-\frac{1}{3}} \cdot dx$

Avem $m = 1$

$n = 2$

$p = -\frac{1}{3} \qquad \frac{m+1}{n} = \frac{1+1}{2} = 1 \in Z$

Facem substitutia $ax^n + b = t^\alpha$

$\Rightarrow x^2 + 2 = t^3 \quad \Rightarrow t = \sqrt[3]{x^2 + 2} \quad \Rightarrow x^2 = t^3 - 2$

$\Rightarrow 2x \cdot dx = 3t^2 \cdot dt \quad \Rightarrow dx = \frac{3t^2}{2x} \cdot dt$ si obtinem

$$F = \int \frac{x \cdot 3t^2 \cdot dt}{\sqrt[3]{x^2 + 2} \cdot 2x} = \int \frac{3t \cdot dt}{2} = \frac{3}{2}\int t \cdot dt = \frac{3}{2} \cdot \frac{t^2}{2} + c = \frac{3}{2} \frac{\sqrt[3]{\left(x^2 + 2\right)^2}}{2} = \frac{3}{4}\left(x^2 + 2\right)^{\frac{2}{3}} + c$$

Exemplul 6.

Se se calculeze primitiva $F = \int \frac{dx}{(1+x)\sqrt{x}} = \int x^{-\frac{1}{2}}(1+x)^{-1} \cdot dx$

Avem $p \in Z$, deci suntem in cazul 1.

Consideram $x = t^r$, unde $r = cmmmc(2,1) = 2$

$\Rightarrow x = t^2 \quad \Rightarrow dx = 2t \cdot dt$ si obtinem

$$F = \int \frac{2t}{\left(1 + t^2\right)t} = \int \frac{2}{1 + t^2} = 2\int \frac{1}{1 + t^2} = 2 \cdot arctg(t) = 2 \cdot arctg\left(\sqrt{x}\right)$$